Lecture 9

Basic Dividers

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Required Reading

Parhami,

Chapter 13 Basic Division Schemes

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Notation

z	Dividend	$z_{2k-1}z_{2k-2} \ldots z_2\ z_1\ z_0$
d	Divisor	$d_{k-1}d_{k-2} \ldots d_1\ d_0$
q	Quotient	$q_{k-1}q_{k-2} \ldots q_1\ q_0$
s	Remainder ($s = z - dq$)	$s_{k-1}s_{k-2} \ldots s_1\ s_0$

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Basic Equations of Division

$$z = d\,q + s$$

$$|s| < |d|$$

$$\text{sign}(s) = \text{sign}(z)$$

$z > 0$: $0 \le s < |d|$

$z < 0$: $-|d| < s \le 0$

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Unsigned Integer Division Overflow

Condition for no overflow:

$$z = q\,d + s < (2^k-1)\,d + d = d\,2^k$$

$$z = z_H\,2^k + z_L < d\,2^k$$

$$z_H < d$$

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Sequential Integer Division Basic Equations

$$s^{(0)} = z$$

$$s^{(j)} = 2\,s^{(j-1)} - q_{k-j}\,(2^k\,d)$$

$$s^{(k)} = 2^k\,s$$

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Sequential Integer Division Justification

$s^{(1)} = 2\,z - q_{k-1}\,(2^k\,d)$
$s^{(2)} = 2(2\,z - q_{k-1}\,(2^k\,d)) - q_{k-2}\,(2^k\,d)$
$s^{(3)} = 2(2(2\,z - q_{k-1}\,(2^k\,d)) - q_{k-2}\,(2^k\,d)) - q_{k-3}\,(2^k\,d)$

. . . . . .

$s^{(k)} = 2(\ldots 2(2(2\,z - q_{k-1}\,(2^k\,d)) - q_{k-2}\,(2^k\,d)) - q_{k-3}\,(2^k\,d) \ldots$
$- q_0\,(2^k\,d) =$
$= 2^k\,z - (2^k\,d)\,(q_{k-1}\,2^{k-1} + q_{k-2}\,2^{k-2} + q_{k-3}\,2^{k-3} + \ldots + q_0 2^0) =$
$= 2^k\,z - (2^k\,d)\,q = 2^k\,(z - d\,q) = 2^k\,s$

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Fig. 13.2 Examples of sequential division with integer and fractional operands.

Integer division			Fractional division		
z	0 1 1 1 0 1 0 1		z_{frac}	. 0 1 1 1 0 1 0 1	
$2^4 d$	1 0 1 0		d_{frac}	. 1 0 1 0	
$s^{(0)}$	0 1 1 1 0 1 0 1		$s^{(0)}$	. 0 1 1 1 0 1 0 1	
$2s^{(0)}$	0 1 1 1 0 1 0 1		$2s^{(0)}$	0 . 1 1 1 0 1 0 1	
$-q_3\,2^4 d$	1 0 1 0	$\{q_3 = 1\}$	$-q_{-1} d$	. 1 0 1 0	$\{q_{-1} = 1\}$
$s^{(1)}$	0 1 0 0 1 0 1		$s^{(1)}$	. 0 1 0 0 1 0 1	
$2s^{(1)}$	0 1 0 0 1 0 1		$2s^{(1)}$	0 . 1 0 0 1 0 1	
$-q_2\,2^4 d$	0 0 0 0	$\{q_2 = 0\}$	$-q_{-2} d$	. 0 0 0 0	$\{q_{-2} = 0\}$
$s^{(2)}$	1 0 0 1 0 1		$s^{(2)}$	. 1 0 0 1 0 1	
$2s^{(2)}$	1 0 0 1 0 1		$2s^{(2)}$	1 . 0 0 1 0 1	
$-q_1\,2^4 d$	1 0 1 0	$\{q_1 = 1\}$	$-q_{-3} d$	. 1 0 1 0	$\{q_{-3} = 1\}$
$s^{(3)}$	1 0 0 0 1		$s^{(3)}$	. 1 0 0 0 1	
$2s^{(3)}$	1 0 0 0 1		$2s^{(3)}$	1 . 0 0 0 1	
$-q_0\,2^4 d$	1 0 1 0	$\{q_0 = 1\}$	$-q_{-1} d$	. 1 0 1 0	$\{q_{-4} = 1\}$
$s^{(4)}$	0 1 1 1		$s^{(4)}$	. 0 1 1 1	
s	0 1 1 1		s_{frac}	0 . 0 0 0 0 0 1 1 1	
q	1 0 1 1		q_{frac}	. 1 0 1 1	

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Unsigned Fractional Division

z_{frac}	Dividend	$.z_{-1}z_{-2}\,.\,.\,.\,z_{-(2k-1)}z_{-2k}$
d_{frac}	Divisor	$.d_{-1}d_{-2}\,.\,.\,.\,d_{-(k-1)}\,d_{-k}$
q_{frac}	Quotient	$.q_{-1}q_{-2}\,.\,.\,.\,q_{-(k-1)}\,q_{-k}$
s_{frac}	Remainder	$\underbrace{.000\ldots0}_{k\ \text{bits}}s_{-(k+1)}\,.\,.\,.\,s_{-(2k-1)}\,s_{-2k}$

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Integer vs. Fractional Division

For Integers:

$$z = q\,d + s \quad \Big| \cdot 2^{-2k}$$

$$z\,2^{-2k} = (q\,2^{-k})\,(d\,2^{-k}) + s\,(2^{-2k})$$

For Fractions:

$$z_{frac} = q_{frac}\,d_{frac} + s_{frac}$$

where

$z_{frac} = z\,2^{-2k}$ $\quad q_{frac} = q\,2^{-k}$

$d_{frac} = d\,2^{-k}$ $\quad s_{frac} = s\,2^{-2k}$

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Unsigned Fractional Division Overflow

Condition for no overflow:

$$z_{frac} < d_{frac}$$

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Sequential Fractional Division Basic Equations

$$s^{(0)} = z_{frac}$$

$$s^{(j)} = 2\,s^{(j-1)} - q_{-j}\,d_{frac}$$

$$2^k \cdot s_{frac} = s^{(k)}$$

$$s_{frac} = 2^{-k} \cdot s^{(k)}$$

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Sequential Fractional Division Justification

$s^{(1)} = 2\, z_{frac} - q_{-1}\, d_{frac}$
$s^{(2)} = 2(2\, z_{frac} - q_{-1}\, d_{frac}) - q_{-2}\, d_{frac}$
$s^{(3)} = 2(2(2\, z_{frac} - q_{-1}\, d_{frac}) - q_{-2}\, d_{frac}) - q_{-3}\, d_{frac}$

. . . . . .

$s^{(k)} = 2(. . . 2(2(2\, z_{frac} - q_{-1}\, d_{frac}) - q_{-2}\, d_{frac}) - q_{-3}\, d_{frac} . . .$
$- q_{-k}\, d_{frac} =$
$= 2^k\, z_{frac} - d_{frac}\, (q_{-1}\, 2^{k-1} + q_{-2}\, 2^{k-2} + q_{-3}\, 2^{k-3} + \ldots + q_{-k} 2^0) =$
$= 2^k\, z_{frac} - d_{frac}\, 2^k (q_{-1}\, 2^{-1} + q_{-2}\, 2^{-2} + q_{-3}\, 2^{-3} + \ldots + q_{-k} 2^{-k}) =$
$= 2^k\, z_{frac} - (2^k\, d_{frac})\, q_{frac} = 2^k\, (z_{frac} - d_{frac}\, q_{frac}) = 2^k\, s_{frac}$

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Restoring Unsigned Integer Division

$s^{(0)} = z$

for $j = 1$ to k

if $2\, s^{(j-1)} - 2^k\, d > 0$
$q_{k-j} = 1$
$s^{(j)} = 2\, s^{(j-1)} - 2^k\, d$
else
$q_{k-j} = 0$
$s^{(j)} = 2\, s^{(j-1)}$

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Fig. 13.5 Shift/subtract sequential restoring divider.

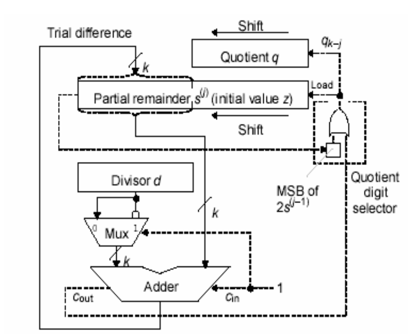

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Fig. 13.6 Example of restoring unsigned division.

z	0 1 1 1	0 1 0 1	No overflow, since: $(0111)_{two} < (1010)_{two}$
$2^4 d$	0 1 0 1 0		
$-2^4 d$	1 0 1 1 0		
$s^{(0)}$	0 0 1 1 1	0 1 0 1	
$2s^{(0)}$	0 1 1 1 0	1 0 1	
$+(-2^4 d)$	1 0 1 1 0		
$s^{(1)}$	0 0 1 0 0	1 0 1	Positive, so set $q_3 = 1$
$2s^{(1)}$	0 1 0 0 1	0 1	
$+(-2^4 d)$	1 0 1 1 0		
$s^{(2)}$	1 1 1 1 1	0 1	Negative, so set $q_2 = 0$ and restore
$s^{(2)}=2s^{(1)}$	0 1 0 0 1	0 1	
$2s^{(2)}$	1 0 0 1 0	1	
$+(-2^4 d)$	1 0 1 1 0		
$s^{(3)}$	0 1 0 0 0	1	Positive, so set $q_1 = 1$
$2s^{(3)}$	1 0 0 0 1		
$+(-2^4 d)$	1 0 1 1 0		
$s^{(4)}$	0 0 1 1 1		Positive, so set $q_0 = 1$
s		0 1 1 1	
q		1 0 1 1	

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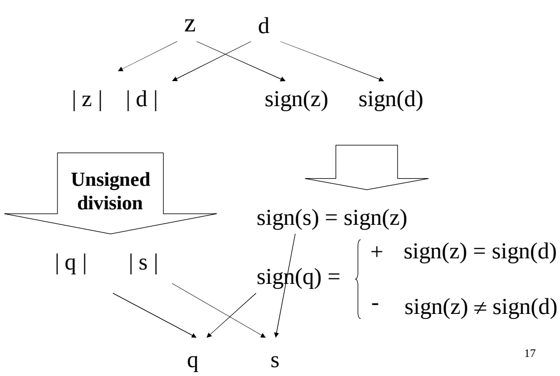
Restoring Signed Integer Division



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Non-Restoring Unsigned Integer Division

$s^{(1)} = z - 2^4 d$
for j = 2 to k
  if $s^{(j-1)} \geq 0$
    $q_{k-(j-1)} = 1$
    $s^{(j)} = 2\, s^{(j-1)} - 2^k\, d$
  else
    $q_{k-(j-1)} = 0$
    $s^{(j)} = 2\, s^{(j-1)} + 2^k\, d$
end for
if $s^{(k)} \geq 0$
  $q_0 = 1$
else
  $q_0 = 0$

if $s^{(k)} < 0$
Correction step

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Fig. 13.7 Example of nonrestoring unsigned division.

z	0 1 1 1	0 1 0 1	No overflow, since: $(0111)_{two} < (1010)_{two}$
2^4d	0 1 0 1 0		
-2^4d	1 0 1 1 0		
$s^{(0)}$	0 0 1 1 1	0 1 0 1	
$2s^{(0)}$	0 1 1 1 0	1 0 1	Positive,
$+(-2^4d)$	1 0 1 1 0		so subtract
$s^{(1)}$	0 0 1 0 0	1 0 1	
$2s^{(1)}$	0 1 0 0 1	0 1	Positive, so set $q_3 = 1$
$+(-2^4d)$	1 0 1 1 0		and subtract
$s^{(2)}$	1 1 1 1 1	0 1	
$2s^{(2)}$	1 1 1 1 0	1	Negative, so set $q_2 = 0$
$+2^4d$	0 1 0 1 0		and add
$s^{(3)}$	0 1 0 0 0	1	
$2s^{(3)}$	1 0 0 0 1		Positive, so set $q_1 = 1$
$+(-2^4d)$	1 0 1 1 0		and subtract
$s^{(4)}$	0 0 1 1 1		Positive, so set $q_0 = 1$
s		0 1 1 1	
q		1 0 1 1	

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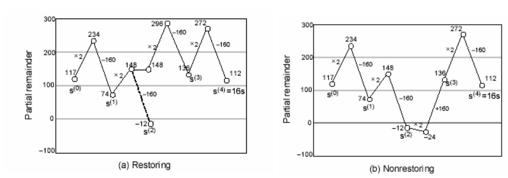
Fig. 13.8 Partial remainder variations for restoring and nonrestoring division.

Example
$(0\,1\,1\,1\;\;0\,1\,0\,1)_{two}$ / $(1\,0\,1\,0)_{two}$
$(117)_{ten}$ / $(10)_{ten}$



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Non-Restoring Unsigned Integer Division

Justification

Restoring division

$s^{(j)} = 2\, s^{(j-1)}$

$s^{(j+1)} = 2\, s^{(j)} - 2^k d =$
$= 4\, s^{(j-1)} - 2^k d$

Non-Restoring division

$s^{(j)} = 2\, s^{(j-1)} - 2^k d$

$s^{(j+1)} = 2\, s^{(j)} + 2^k d =$
$= 2\,(2\, s^{(j-1)} - 2^k d) + 2^k d =$
$= 4\, s^{(j-1)} - 2^k d$

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Non-Restoring Signed Integer Division

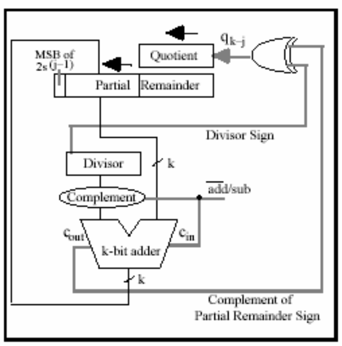
Correction step

z = q d + s

z = (q-1) d + (s+d)
z = q’ d + s’

z = (q+1) d + (s-d)
z = q” d + s”

Fig. 13.10 Shift-subtract sequential nonrestoring divider.



Non-Restoring Signed Integer Division

$s^{(0)} = z$
for j = 1 to k
  if sign($s^{(j-1)}$) == sign(d)
    $q_{k-j} = 1$
    $s^{(j)} = 2\, s^{(j-1)} - 2^k d = 2\, s^{(j-1)} - q_{k-j}\,(2^k d)$
  else
    $q_{k-j} = -1$
    $s^{(j)} = 2\, s^{(j-1)} + 2^k d = 2\, s^{(j-1)} - q_{k-j}\,(2^k d)$

Correction_step
q = BSD_2’s_comp_conversion(q);

Fig. 13.9 Example of nonrestoring signed division.

z	0 0 1 0 0 0 0 1	Dividend = $(33)_{ten}$
2^4d	1 1 0 0 1	Divisor = $(-7)_{ten}$
-2^4d	0 0 1 1 1	
$s^{(0)}$	0 0 0 1 0 0 0 0 1	
$2s^{(0)}$	0 0 1 0 0 0 0 1	sign($s^{(0)}$) • sign(d),
$+2^4d$	1 1 0 0 1	so set $q_3 = -1$ and add
$s^{(1)}$	1 1 1 0 1 0 0 1	
$2s^{(1)}$	1 1 0 1 0 0 1	sign($s^{(1)}$) = sign(d),
$+(-2^4d)$	0 0 1 1 1	so set $q_2 = 1$ and sub
$s^{(2)}$	0 0 0 0 1 0 1	
$2s^{(2)}$	0 0 0 1 0 1	sign($s^{(2)}$) • sign(d),
$+2^4d$	1 1 0 0 1	so set $q_1 = -1$ and add
$+(-2^4d)$	1 0 1 1 0	
$s^{(3)}$	1 1 0 1 1 1	
$2s^{(3)}$	1 0 1 1 1	sign($s^{(3)}$) = sign(d),
$+(-2^4d)$	0 0 1 1 1	so set $q_0 = 1$ and sub
$s^{(4)}$	1 1 1 1 0	sign($s^{(4)}$) • sign(z)
$+(-2^4d)$	0 0 1 1 1	Corrective subtraction
$s^{(4)}$	0 0 1 0 1	
s	0 1 0 1	Remainder = $(5)_{ten}$
q	-1 1 -1 1	Uncorrected BSD form
$q_{2's\text{-}compl}$	1 1 0 0	Corrected $q = (-4)_{ten}$

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BSD → 2's Complement Conversion

$$q = (q_{k-1}\ q_{k-2} \ldots q_1\ q_0)_{BSD} =$$

$$= (\overline{p_{k-1}}\ p_{k-2} \ldots p_1\ p_0\ 1)_{2\text{'s complement}}$$

where

q_i	p_i
-1	0
1	1

Example:

q_{BSD}	1 -1 1 1
p	1 0 1 1
$q_{2\text{'s comp}}$	0 0 1 1 1 = 0 1 1 1

no overflow if $p_{k-2} = \overline{p_{k-1}}$ $(q_{k-1} \neq q_{k-2})$

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