Lecture 7

Tree and Array Multipliers

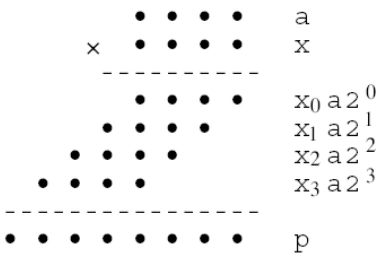
❝At least one good reason
for studying multiplication and division is that
there is an infinite number of ways
of performing these operations
and hence there is an infinite number of PhDs
(or expenses-paid visits to conferences in the USA)
to be won from inventing new forms of multiplier.❞

Alan Clements
The Principles of Computer Hardware, 1986

Notation

a	Multiplicand	$a_{k-1}a_{k-2} \ldots a_1\, a_0$
x	Multiplier	$x_{k-1}x_{k-2} \ldots x_1\, x_0$
p	Product (a · x)	$p_{2k-1}p_{2k-2} \ldots p_2\, p_1\, p_0$

Multiplication of two 4-bit unsigned binary numbers in dot notation



Basic Multiplication Equations

$$p = a \cdot x \qquad x = \sum_{i=0}^{k-1} x_i \cdot 2^i$$

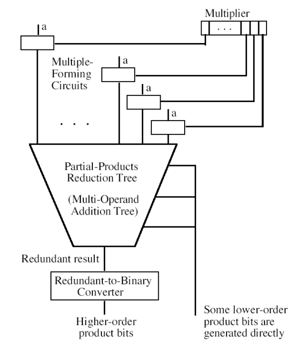
$$p = a \cdot x = \sum_{i=0}^{k-1} a \cdot x_i \cdot 2^i =$$

$$= x_0 a 2^0 + x_1 a 2^1 + x_2 a 2^2 + \ldots + x_{k-1} a 2^{k-1}$$

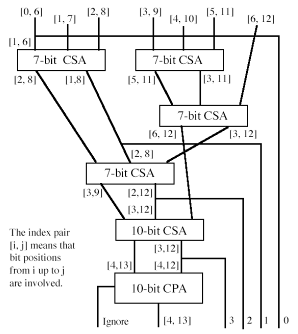
Unsigned Multiplication

						a_4	a_3	a_2	a_1	a_0
					x	x_4	x_3	x_2	x_1	x_0
$ax_0\ 2^0$						a_4x_0	a_3x_0	a_2x_0	a_1x_0	a_0x_0
$ax_1\ 2^1$		+			a_4x_1	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
$ax_2\ 2^2$				a_4x_2	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
$ax_3\ 2^3$			a_4x_3	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
$ax_4\ 2^4$		a_4x_4	a_3x_4	a_2x_4	a_1x_4	a_0x_4				
	p_9	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0

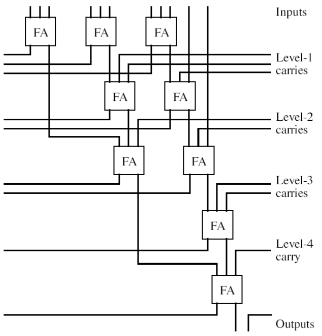
Full tree multiplier - general structure



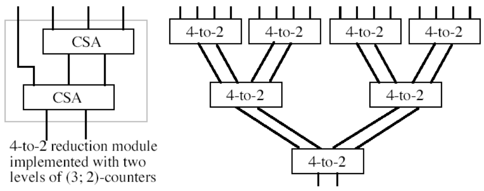
7 x 7 tree multiplier



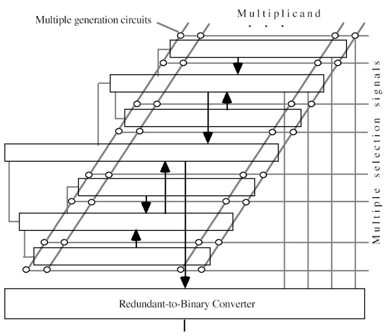
A slice of a balanced-delay tree for 11 inputs



Tree multiplier with a more regular structure



Layout of a multiplier based on 4-to-2 reduction modules



2's Complement Multiplication (1)

	-2^4	2^3	2^2	2^1	2^0
	a_4	a_3	a_2	a_1	a_0
x	**x_4**	x_3	x_2	x_1	x_0

≡

	2^4	2^3	2^2	2^1	2^0
	$-a_4$	a_3	a_2	a_1	a_0
x	**$-x_4$**	x_3	x_2	x_1	x_0

2’s Complement Multiplication (2)

						$-a_4$	a_3	a_2	a_1	a_0
					x	$-x_4$	x_3	x_2	x_1	x_0
						$-a_4x_0$	a_3x_0	a_2x_0	a_1x_0	a_0x_0
+					$-a_4x_1$	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
				$-a_4x_2$	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
			$-a_4x_3$	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
		a_4x_4	$-a_3x_4$	$-a_2x_4$	$-a_1x_4$	$-a_0x_4$				
	$-p_9$	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

2’s Complement Multiplication (3)

$-p_9$	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

$\Leftrightarrow \; \equiv$

p_9	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
-2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

2’s Complement Multiplication (4)

$$\overline{z} = 1 - z$$

$$z = 1 - \overline{z}$$

$$- a_j x_i = - a_j (1 - \overline{x_i}) = a_j \overline{x_i} - a_j = a_j \overline{x_i} + a_j - 2 a_j$$

$$- a_j x_i = - (1 - \overline{a_j}) x_i = \overline{a_j} x_i - x_i = \overline{a_j} x_i + x_i - 2 x_i$$

$$- a_j x_i = - (1 - \overline{a_j x_i}) = \overline{a_j x_i} - 1 = \overline{a_j x_i} + 1 - 2$$

$$-a_j = - (1 - \overline{a_j}) = \overline{a_j} - 1 = \overline{a_j} + 1 - 2$$

$$-x_i = - (1 - \overline{x_i}) = \overline{x_i} - 1 = \overline{x_i} + 1 - 2$$

					$-a_4x_0$
				$-a_4x_1$	
			$-a_4x_2$		
+		$-a_4x_3$			
				$-a_4$	$a_4\overline{x}_0$
			$-a_4$	$a_4\overline{x}_1$	a_4
		$-a_4$	$a_4\overline{x}_2$	a_4	
	$-a_4$	$a_4\overline{x}_3$	a_4		
		a_4			
	$\overline{a}_4$	**$a_4\overline{x}_3$**	**$a_4\overline{x}_2$**	**$a_4\overline{x}_1$**	**$a_4\overline{x}_0$**
	-1				**a_4**

+		$-a_3x_4$	$-a_2x_4$	$-a_1x_4$	$-a_0x_4$
				$-x_4$	$\overline{a}_0x_4$
			$-x_4$	$\overline{a}_1x_4$	x_4
		$-x_4$	$\overline{a}_2x_4$	x_4	
	$-x_4$	$\overline{a}_3x_4$	x_4		
		x_4			
	$\overline{x}_4$	**$\overline{a}_3x_4$**	**$\overline{a}_2x_4$**	**$\overline{a}_1x_4$**	**$\overline{a}_0x_4$**
	-1				**x_4**

2^9	2^8	2^7	2^6	2^5	2^4
	$\overline{a}_4$	**$a_4\overline{x}_3$**	**$a_4\overline{x}_2$**	**$a_4\overline{x}_1$**	**$a_4\overline{x}_0$**
	-1				**a_4**
	$\overline{x}_4$	**$\overline{a}_3x_4$**	**$\overline{a}_2x_4$**	**$\overline{a}_1x_4$**	**$\overline{a}_0x_4$**
	-1				**x_4**
-1	**$\overline{a}_4$**	**$a_4\overline{x}_3$**	**$a_4\overline{x}_2$**	**$a_4\overline{x}_1$**	**$a_4\overline{x}_0$**
	$\overline{x}_4$	**$\overline{a}_3x_4$**	**$\overline{a}_2x_4$**	**$\overline{a}_1x_4$**	**$\overline{a}_0x_4$**
					a_4
					x_4
1	**$\overline{a}_4$**	**$a_4\overline{x}_3$**	**$a_4\overline{x}_2$**	**$a_4\overline{x}_1$**	**$a_4\overline{x}_0$**
	$\overline{x}_4$	**$\overline{a}_3x_4$**	**$\overline{a}_2x_4$**	**$\overline{a}_1x_4$**	**$\overline{a}_0x_4$**
					a_4
-2^9					**x_4**

Baugh-Wooley 2's Complement Multiplier

						$-a_4$	a_3	a_2	a_1	a_0
					x	$-x_4$	x_3	x_2	x_1	x_0
						$a_4\overline{x_0}$	a_3x_0	a_2x_0	a_1x_0	a_0x_0
+					$a_4\overline{x_1}$	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
				$a_4\overline{x_2}$	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
			$a_4\overline{x_3}$	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
		a_4x_4	$\overline{a_3}x_4$	$\overline{a_2}x_4$	$\overline{a_1}x_4$	$\overline{a_0}x_4$				
		$\overline{a_4}$				a_4				
	1	$\overline{x_4}$				x_4				
	p_9	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
	-2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

					$-a_4x_0$
				$-a_4x_1$	
			$-a_4x_2$		
+		$-a_4x_3$			
				-1	$\overline{a_4x_0}$
			-1	$\overline{a_4x_1}$	1
		-1	$\overline{a_4x_2}$	1	
	-1	$\overline{a_4x_3}$	1		
		1			
		$\overline{a_4x_3}$	**$\overline{a_4x_2}$**	**$\overline{a_4x_1}$**	**$\overline{a_4x_0}$**
	-1				**1**

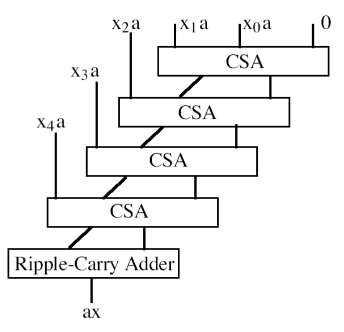
+		$-a_3x_4$	$-a_2x_4$	$-a_1x_4$	$-a_0x_4$
				-1	$\overline{a_0x_4}$
			-1	$\overline{a_1x_4}$	1
		-1	$\overline{a_2x_4}$	1	
	-1	$\overline{a_3x_4}$	1		
		1			
		$\overline{a_3x_4}$	**$\overline{a_2x_4}$**	**$\overline{a_1x_4}$**	**$\overline{a_0x_4}$**
	-1				**1**

2^9	2^8	2^7	2^6	2^5	2^4
		$\overline{a_4x_3}$	$\overline{a_4x_2}$	$\overline{a_4x_1}$	$\overline{a_4x_0}$
	-1				1
		$\overline{a_3x_4}$	$\overline{a_2x_4}$	$\overline{a_1x_4}$	$\overline{a_0x_4}$
	-1				1
		$\overline{a_4x_3}$	$\overline{a_4x_2}$	$\overline{a_4x_1}$	$\overline{a_4x_0}$
		$\overline{a_3x_4}$	$\overline{a_2x_4}$	$\overline{a_1x_4}$	$\overline{a_0x_4}$
-1				1	
		$\overline{a_4x_3}$	$\overline{a_4x_2}$	$\overline{a_4x_1}$	$\overline{a_4x_0}$
		$\overline{a_3x_4}$	$\overline{a_2x_4}$	$\overline{a_1x_4}$	$\overline{a_0x_4}$
1				1	
-2^9					

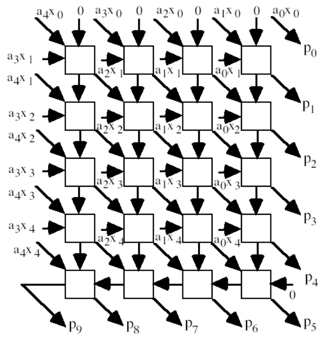
Modified Baugh-Wooley Multiplier

						$-a_4$	a_3	a_2	a_1	a_0
					x	$-x_4$	x_3	x_2	x_1	x_0
						$\overline{a_4x_0}$	a_3x_0	a_2x_0	a_1x_0	a_0x_0
+					$\overline{a_4x_1}$	a_3x_1	a_2x_1	a_1x_1	a_0x_1	
				$\overline{a_4x_2}$	a_3x_2	a_2x_2	a_1x_2	a_0x_2		
			$\overline{a_4x_3}$	a_3x_3	a_2x_3	a_1x_3	a_0x_3			
		a_4x_4	$\overline{a_3x_4}$	$\overline{a_2x_4}$	$\overline{a_1x_4}$	$\overline{a_0x_4}$				
	1				1					
	p_9	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
	-2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

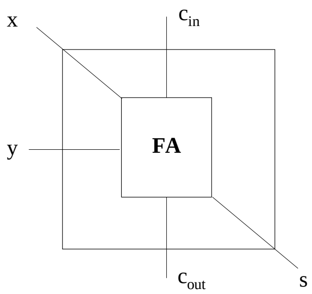
Basic array multiplier



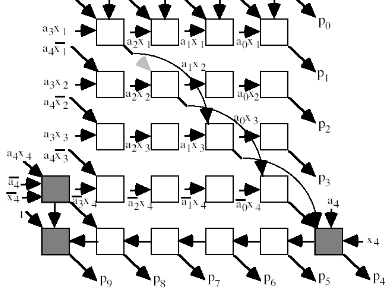
5 x 5 Array Multiplier



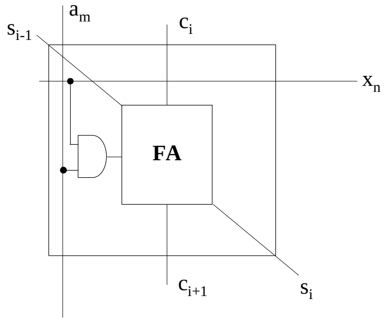
Array Multiplier - Basic Cell



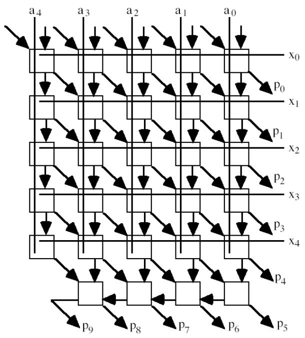
Modifications in a 5 x 5 multiplier



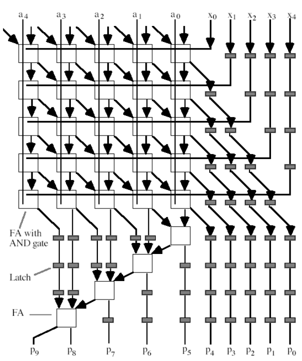
Array Multiplier – Modified Basic Cell



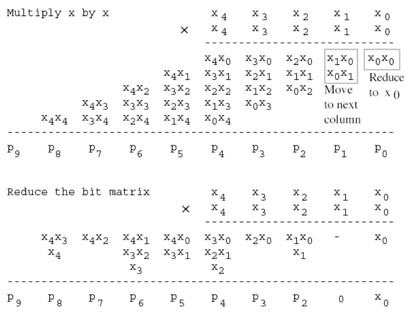
5 x 5 Array Multiplier with modified cells



Pipelined 5 x 5 Multiplier



Optimizations for Squaring (1)



Optimizations for Squaring (2)

$$\begin{array}{r} x_i\, x_j \\ x_j\, x_i \\ \hline x_i\, x_j \quad \end{array} \qquad x_i\, x_j + x_i\, x_j = 2\, x_i\, x_j$$

$$x_i\, x_i = x_i$$

$$\begin{array}{r} x_i\, x_j \\ x_i \\ \hline x_i\, x_j \quad x_i\, \overline{x_j} \end{array} \qquad x_i\, x_j + x_i = 2\, x_i\, x_j - x_i\, x_j + x_i =$$

$$= 2\, x_i\, x_j + x_i\, (1 - x_j) =$$

$$= 2\, x_i\, x_j + x_i\, \overline{x_j}$$

Squaring Using Look-Up Tables

for relatively small values k

input=a	output=a^2
0	0
1	1
2	4
3	9
4	16
	. . .
i	i^2
	. . .
2^k-1	$(2^k-1)^2$

2^k words 2k-bit each

Multiplication Using Squaring

$$a \cdot x = \frac{(a+x)^2 - (a-x)^2}{4}$$